Online resit exam — Functional Analysis (WBMA033-05)

Wednesday 23 June 2021, 8.30h–11.30h CEST (plus 30 minutes for uploading)

University of Groningen

Instructions

1. Only references to the lecture notes and slides are allowed. References to other sources are *not* allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.
4. Write both your name and student number on the answer sheets!
5. This exam comes in two versions. Both versions consist of five problems of equal difficulty.

   Make version 1 if your student number is odd.

   Make version 2 if your student number is even.

   For example, if your student number is 1277456, which is even, then you have to make version 2.
6. Please submit your work as a single PDF file.

Version 1 (for odd student numbers)

Problem 1 (5 + 10 = 15 points)

Consider the following linear space:

$$X = \left\{ x = (x_1, x_2, x_3, \dots) \,:\, x_k \in \mathbb{K} \text{ and } \sum_{k=1}^{\infty} 3^k |x_k| < \infty \right\}.$$

This space can be equipped with the following norms:

$$\|x\|_1 = \sum_{k=1}^{\infty} k^2 |x_k| \quad \text{and} \quad \|x\|_2 = \sum_{k=1}^{\infty} 3^k |x_k|.$$

(a) Show that there exists $C > 0$ such that $\|x\|_1 \leq C\|x\|_2$ for all $x \in X$.

(b) Are the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ equivalent?

Problem 2 (5 + 10 + 10 = 25 points)

Let X be a Hilbert space over $\mathbb{R}$ such that $\dim X \geq 2$. Recall that the space $B(X)$ becomes a Banach space in its own right when equipped with the operator norm. Consider the following linear operator:

$$T : B(X) \to B(X), \quad TA = A^*.$$

(a) Compute the operator norm of T.

(b) Assume that the vectors $e_1, e_2 \in X$ are orthonormal. Show that the linear operators $A_1, A_2 \in B(X)$ defined by

$$A_1 x = 3(x, e_1)e_1 + 5(x, e_2)e_2 \quad \text{and} \quad A_2 x = (x, e_1)e_2 - (x, e_2)e_1.$$

are eigenvectors of T. What are the corresponding eigenvalues?

(c) Determine $\rho(T)$ and $\sigma(T)$ by explicitly computing $(T - \lambda)^{-1}$.
Hint: for $B \in B(X)$ take the adjoint of the equation $A^* - \lambda A = B$ to get a second equation and solve for A in terms of B and λ.

Problem 3 (15 points)

Let X be a normed linear space, and assume that (x_k) is a sequence in X such that

$$\sum_{k=1}^{\infty} |f(x_k)| < \infty \quad \text{for all} \quad f \in X'.$$

Prove that

$$\sup \left\{ \sum_{k=1}^{\infty} |f(x_k)| \,:\, f \in X',\ \|f\| \leq 1 \right\} < \infty.$$

Hint: consider the operators given by

$$T_n : X' \to \ell^1, \quad T_n f = (f(x_1), \dots, f(x_n), 0, 0, 0, \dots).$$

Problem 4 (5 + 5 + 5 + 5 + 5 = 25 points)

Let X be a Hilbert space over $\mathbb{C}$, and assume that $S \in B(X)$ is selfadjoint.

(a) Show that for $b \in \mathbb{R}$ we have

$$\|(S - bi)x\|^2 = \|Sx\|^2 + b^2\|x\|^2 \quad \text{for all} \quad x \in X.$$

(b) Prove that $S + i$ is invertible.

Next, consider the operator $U = (S - i)(S + i)^{-1}$.

(c) Show that $\|Ux\| = \|x\|$ for all $x \in X$.
Hint: observe that part (a) implies that $\|(S - i)x\| = \|(S + i)x\|$.

(d) Show that $(Ux, Uy) = (x, y)$ for all $x, y \in X$.

(e) Prove that U is unitary.

Problem 5 (10 points)

Equip the linear space $X = \mathcal{C}([0, 2\pi], \mathbb{C})$ with the following norm:

$$\|f\| = \int_0^{2\pi} |f(x)|\, dx, \qquad f \in X.$$

Let $g(x) = \sin(x)$. Prove that there exists a functional $\varphi \in X'$ such that

$$\varphi(g) = 2 - 4i \qquad \text{and} \qquad \|\varphi\| = \frac{\sqrt{5}}{2}.$$

End of test ("version 1", 90 points)

Version 2 (for odd student numbers)

Problem 1 (5 + 10 = 15 points)

Consider the following linear space:

$$X = \left\{ x = (x_1, x_2, x_3, \dots) \; : \; x_k \in \mathbb{K} \text{ and } \sum_{k=1}^{\infty} 5^k |x_k| < \infty \right\}.$$

This space can be equipped with the following norms:

$$\|x\|_1 = \sum_{k=1}^{\infty} k^2 |x_k| \quad \text{and} \quad \|x\|_2 = \sum_{k=1}^{\infty} 5^k |x_k|.$$

(a) Show that there exists $C > 0$ such that $\|x\|_1 \leq C\|x\|_2$ for all $x \in X$.

(b) Are the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ equivalent?

Problem 2 (5 + 10 + 10 = 25 points)

Let X be a Hilbert space over $\mathbb{R}$ such that $\dim X \geq 2$. Recall that the space $B(X)$ becomes a Banach space in its own right when equipped with the operator norm. Consider the following linear operator:

$$T : B(X) \to B(X), \quad TA = A^*.$$

(a) Compute the operator norm of T.

(b) Assume that the vectors $e_1, e_2 \in X$ are orthonormal. Show that the linear operators $A_1, A_2 \in B(X)$ defined by

$$A_1 x = 5(x, e_1)e_1 + 7(x, e_2)e_2 \quad \text{and} \quad A_2 x = (x, e_1)e_2 - (x, e_2)e_1.$$

are eigenvectors of T. What are the corresponding eigenvalues?

(c) Determine $\rho(T)$ and $\sigma(T)$ by explicitly computing $(T - \lambda)^{-1}$.
Hint: for $B \in B(X)$ take the adjoint of the equation $A^* - \lambda A = B$ to get a second equation and solve for A in terms of B and λ.

Problem 3 (15 points)

Let X be a normed linear space, and assume that (x_k) is a sequence in X such that

$$\sum_{k=1}^{\infty} |f(x_k)| < \infty \quad \text{for all} \quad f \in X'.$$

Prove that

$$\sup \left\{ \sum_{k=1}^{\infty} |f(x_k)| \; : \; f \in X', \; \|f\| \leq 1 \right\} < \infty.$$

Hint: consider the operators given by

$$T_n : X' \to \ell^1, \quad T_n f = (f(x_1), \dots, f(x_n), 0, 0, 0, \dots).$$

Problem 4 (5 + 5 + 5 + 5 + 5 = 25 points)

Let X be a Hilbert space over $\mathbb{C}$, and assume that $S \in B(X)$ is selfadjoint.

(a) Show that for $b \in \mathbb{R}$ we have

$$\|(S - bi)x\|^2 = \|Sx\|^2 + b^2\|x\|^2 \quad \text{for all} \quad x \in X.$$

(b) Prove that $S + i$ is invertible.

Next, consider the operator $U = (S - i)(S + i)^{-1}$.

(c) Show that $\|Ux\| = \|x\|$ for all $x \in X$.
Hint: observe that part (a) implies that $\|(S - i)x\| = \|(S + i)x\|$.

(d) Show that $(Ux, Uy) = (x, y)$ for all $x, y \in X$.

(e) Prove that U is unitary.

Problem 5 (10 points)

Equip the linear space $X = \mathcal{C}([0, 2\pi], \mathbb{C})$ with the following norm:

$$\|f\| = \int_0^{2\pi} |f(x)|\, dx, \qquad f \in X.$$

Let $g(x) = \cos(x)$. Prove that there exists a functional $\varphi \in X'$ such that

$$\varphi(g) = -2 + 2i \qquad \text{and} \qquad \|\varphi\| = \frac{\sqrt{2}}{2}.$$

End of test ("version 2", 90 points)

Solution of problem 1 (5 + 10 = 15 points)

(a) It is obvious that $k < (\sqrt{3})^k$ for all $k \in \mathbb{N}$. Taking the square of both sides gives that $k^2 < 3^k$ for all $k \in \mathbb{N}$.
(1 point)

This implies that for all $x \in X$ we have

$$\|x\|_1 = \sum_{k=1}^{\infty} k^2 |x_k| \leq \sum_{k=1}^{\infty} 3^k |x_k| = \|x\|_2.$$

(3 points)

Therefore, we can take $C = 1$.
(1 point)

(b) Suppose that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. Then there exists a constant $C' > 0$ such that the inequality $\|x\|_2 \leq C'\|x\|_1$ holds for all $x \in X$.
(3 points)

In particular, this inequality needs to hold for the vectors

$$x^n = (0, \dots, 0, 1, 0, 0, 0 \dots),$$

where the 1 is at the n-th entry. This implies that $3^n \leq C'n^2$ for all $n \in \mathbb{N}$, or, equivalently,

$$1 \leq C' \frac{n^2}{3^n} \quad \text{for all} \quad n \in \mathbb{N}.$$

(3 points)

However, the right hand side tends to zero for $n \to \infty$. Therefore, such a constant C' cannot exist. We conclude that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent.
(4 points)

Solution of problem 2 (5 + 10 + 10 = 25 points)

(a) For any $A \in B(X)$ we have that $\|TA\| = \|A^*\| = \|A\|$.
(2 points)

This implies that the operator norm of T is given by

$$\|T\| = \sup_{A \in B(X), A \neq 0} \frac{\|TA\|}{\|A\|} = 1.$$

(3 points)

(b) The assumption that e_1 and e_2 are orthonormal implies that

$$A_1 e_1 = \begin{cases} 3e_1 & \text{for version 1,} \\ 5e_1 & \text{for version 2,} \end{cases} \quad \text{and} \quad A_2 e_1 = -e_2,$$

which implies that neither A_1 nor A_2 is the zero operator.
(2 points)

Let $u, v \in X$ be arbitrary vectors, and consider the operator $A \in B(X)$ given by $Ax = (x, u)v$. Then for all $x, y \in X$ we have that

$$(Ax, y) = ((x, u)v, y) = (x, u)(v, y) = (x, u)\overline{(y, v)} = (x, (y, v)u) = (x, A^*y),$$

which shows that the adjoint of A given by $A^*x = (x, v)u$.
(4 points)

Since adjoints can be taken term by term, we find that by setting $u = v = e_1$ and $u = v = e_2$ that

$$A_1^* x = \begin{cases} 3(x, e_1)e_1 + 5(x, e_2)e_2 & \text{for version 1,} \\ 5(x, e_1)e_1 + 7(x, e_2)e_2 & \text{for version 2,} \end{cases}$$

which implies that $TA_1 = A_1$. Therefore, A_1 is an eigenvector of T for the eigenvalue $\lambda = 1$.
(2 points)

Likewise, by setting $u = e_2$ and $v = e_1$ (and *vice versa*) we find that

$$A_2^* x = (x, e_2)e_1 - (x, e_1)e_2,$$

which implies that $TA_2 = -A_2$. Therefore, A_2 is an eigenvector of T for the eigenvalue $\lambda = -1$.
(2 points)

(c) Let $B \in B(X)$ be arbitrary, and assume that $(T - \lambda)A = B$, or equivalently,

$$A^* - \lambda A = B.$$

Taking the adjoint of this equation gives

$$A - \lambda A^* = B^*.$$

From the first equation we obtain that $A^* = \lambda A + B$. Substitution in the second equation gives

$$A - \lambda(\lambda A + B) = B^*.$$

(2 points)

Solving for A gives

$$A = \frac{\lambda}{1-\lambda^2} B + \frac{1}{1-\lambda^2} B^*,$$

(2 points)

which shows that for $\lambda \neq \pm 1$ we have that

$$(T-\lambda)^{-1} = \frac{\lambda}{1-\lambda^2} I + \frac{1}{1-\lambda^2} T.$$

(2 points)

Clearly, the right hand side is a linear combination of the bounded operators I and T.
(2 points)

We conclude that $\rho(T) = \mathbb{R} \setminus \{-1, 1\}$ and hence $\sigma(T) = \{-1, 1\}$.
(2 points)

Solution of problem 3 (15 points)

Define the operator

$$T : X' \to \ell^1, \quad T_n f = (f(x_1), \dots, f(x_n), \dots).$$

We claim that for all $f \in X'$ we have

$$\|T_n f - Tf\|_1 = \sum_{k=n+1}^{\infty} |f(x_k)| \to 0 \quad \text{as} \quad n \to \infty.$$

Indeed, since the series $\sum_{k=1}^{\infty} |f(x_k)|$ converges it follows that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n, m \geq N \quad \Rightarrow \quad \sum_{k=n+1}^{m} |f(x_k)| < \varepsilon.$$

Taking $m \to \infty$ gives

$$n \geq N \quad \Rightarrow \quad \sum_{k=n+1}^{\infty} |f(x_k)| < \varepsilon.$$

which proves the claim.
(5 points)

This means that the sequence (T_n) converges *pointwise* to T. By a corollary of the Uniform Boundedness Principle it follows that $T \in B(X', \ell^1)$, which means that there exists a constant $C > 0$ such that

$$\|Tf\| \leq C\|f\| \quad \text{for all} \quad f \in X'.$$

(5 points)

In particular, for any $f \in X'$ with $\|f\| \leq 1$ it follows that

$$\sum_{k=1}^{\infty} |f(x_k)| \leq C.$$

Taking the supremum over all such elements gives

$$\sup \left\{ \sum_{n=1}^{\infty} |f(x_n)| \,:\, f \in X',\, \|f\| \leq 1 \right\} \leq C < \infty.$$

(5 points)

Solution of problem 4 (5 + 5 + 5 + 5 + 5 = 25 points)

(a) Expanding the norm in terms of the innerproduct gives

$$\begin{aligned}\|(S-bi)x\|^2 &= ((S-bi)x,(S-bi)x)\\ &= (Sx,Sx)+(-bix,Sx)+(Sx,-bix)+(-bix,-bix)\\ &= (Sx,Sx)-bi(x,Sx)+bi(Sx,x)+|bi|^2(x,x)\\ &= \|Sx\|^2-bi(Sx,x)+bi(Sx,x)+|b|^2\|x\|^2\\ &= \|Sx\|^2+|b|^2\|x\|^2.\end{aligned}$$

(5 points)

(b) *Method 1.* Since S is selfadjoint we have that $\sigma(S)\subset\mathbb{R}$, which implies that $\mathbb{C}\setminus\mathbb{R}\subset\rho(T)$. In particular, it follows that $-i\in\rho(S)$, which means that $S+i$ is invertible.
(5 points)

Method 2. From part (a) it follows that

$$\|(S+i)x\|^2\geq\|x\|^2 \quad \text{for all} \quad x\in X.$$

The operator S is selfadjoint and thus also normal. Therefore, the inequality above implies that $-i\in\rho(S)$, which means that $S+i$ is invertible.
(5 points)

(c) Observe that part (a) implies that $\|(S-i)x\|=\|(S+i)x\|$. This implies that

$$\|Ux\|=\|(S-i)(S+i)^{-1}x\|=\|(S+i)(S+i)^{-1}x\|=\|x\|.$$

(5 points)

(d) By the polarization identity and part (c) we have

$$(Ux,Uy)=\sum_{k=0}^{3}i^k\|U(x+i^ky)\|^2=\sum_{k=0}^{3}i^k\|x+i^ky\|^2=(x,y).$$

(e) From part (d) we obtain

$$((U^*U-I)x,y)=(U^*Ux,y)-(x,y)=(Ux,Uy)-(x,y)=0 \quad \text{for all} \quad x,y\in X.$$

In particular, taking $y=(U^*U-I)x$ gives

$$\|(U^*U-I)x\|^2=0 \quad \text{for all} \quad x\in X.$$

This implies that $U^*U=I$.
(4 points)

In a similar manner it is shown that $UU^*=I$. We conclude that U is unitary.
(1 point)

Solution of problem 5 (10 points)

Version 1. Define the map

$$\varphi : \text{span}\,\{g\} \to \mathbb{C}, \quad \varphi(\lambda g) = \lambda(2 - 4i).$$

With $\lambda = 1$ we have that $\varphi(g) = 4 - 2i$.
(2 points)

Since $\|g\| = 4$ we have that

$$\|\varphi\| = \sup_{\lambda \neq 0} \frac{|\varphi(\lambda g)|}{\|\lambda g\|} = \sup_{\lambda \neq 0} \frac{|\lambda|\sqrt{20}}{4|\lambda|} = \frac{\sqrt{5}}{2}.$$

(5 points)

Now apply the Hahn-Banach theorem to extend φ to the entire space X while preserving the norm.
(3 points)

Version 2. Almost identical: the only change is that $|4 - 2i| = \sqrt{20}$ has to be replaced by $|-2 + 2i| = \sqrt{8}$.